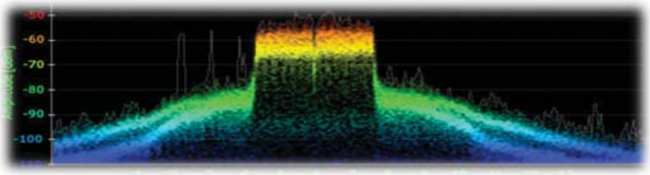
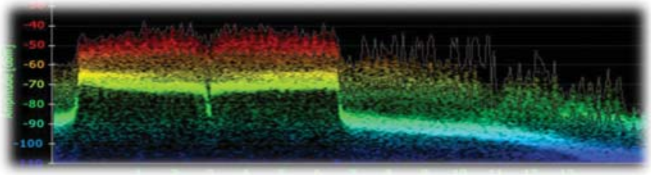

ECS455: Chapter 5

OFDM

5.3 OFDM as Multi-Carrier Transmission

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prapun.com/ecs455

Single-Carrier Digital Transmission

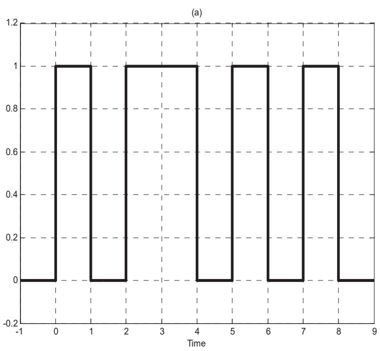
- Baseband:

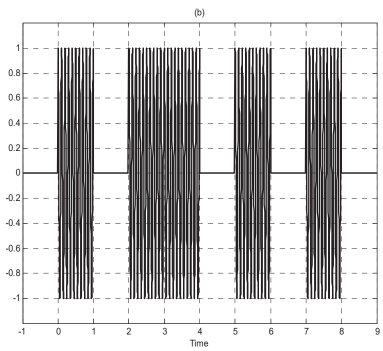
$$s(t)=\sum_{k=0}^{N-1} s_k\, p(t-kT_s)$$

$$p(t)=1_{[0,T_s)}(t)=\begin{cases}1, & t\in[0,T_s)\\ 0, & \text{otherwise.}\end{cases}$$

- Passband:

$$x(t)=\mathrm{Re}\left\{s(t)e^{j2\pi f_c t}\right\}$$

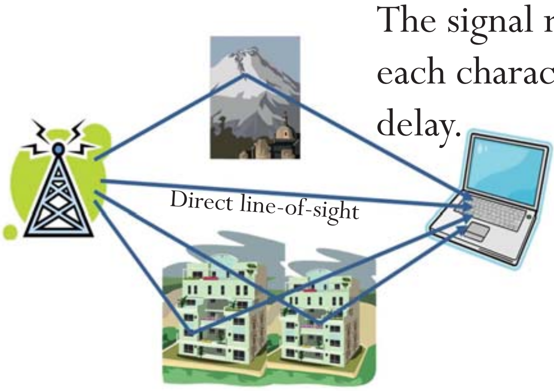




Review: Multipath Propagation

- In a wireless mobile communication system, a transmitted signal propagating through the wireless channel often encounters multiple reflective paths until it reaches the receiver
- We refer to this phenomenon as **multipath propagation** and it causes fluctuation of the amplitude and phase of the received signal.
- We call this fluctuation **multipath fading**.

Wireless Comm. and Multipath Fading



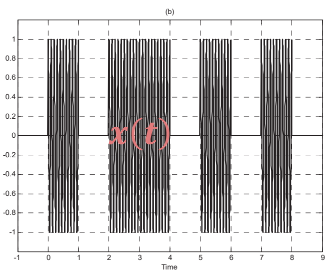
The signal received consists of a number of reflected rays, each characterized by a different amount of attenuation and delay.

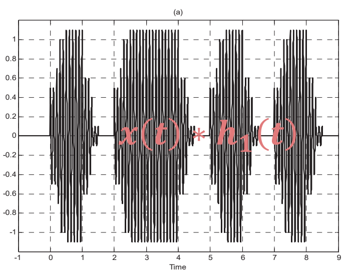
$$y(t)=x(t)*h(t)+n(t)=\sum_{i=0}^{v}\beta_i x(t-\tau_i)+n(t)$$

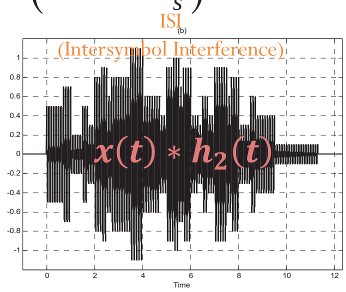
$$h(t)=\sum_{i=0}^{v}\beta_i\delta(t-\tau_i)$$

$$h_1(t)=0.5\delta(t)+0.2\delta(t-0.2T_s)+0.3\delta(t-0.3T_s)+0.1\delta(t-0.5T_s)$$

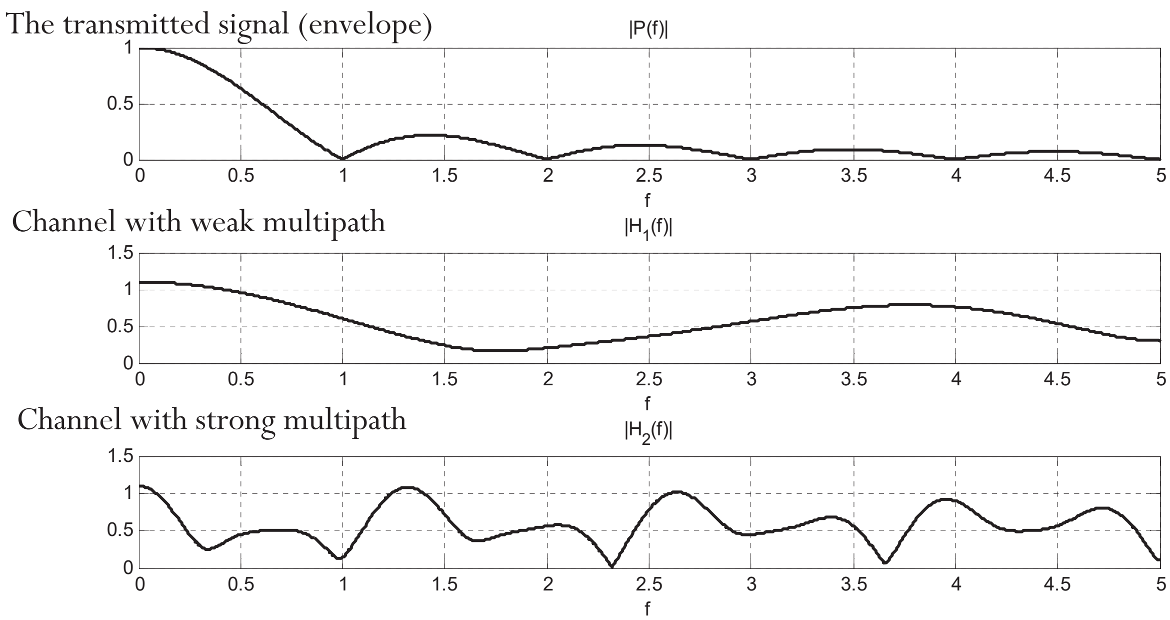
$$h_2(t)=0.5\delta(t)+0.2\delta(t-0.7T_s)+0.3\delta(t-1.5T_s)+0.1\delta(t-2.3T_s)$$







Frequency Domain



Distortionless Channel

Definition 3.18. A channel is called **distortionless** if

$$y(t) = \beta x(t - \tau),$$

where β and τ are constants.

- The channel output has the same "shape" as its input.
- This is the "best" channel we can hope for. Any transmitted signal $x(t)$ will need to travel over some distance before it reaches the receiver. It will be delayed by the propagation time and its power will be attenuated.

τ: propagation delay = distance / c (+ processing delay)

impulse response $= h(t) = \beta\delta(t-\tau)$

freq. response $= H(f) = \beta e^{-j2\pi(\tau)(f)}$

From the Fourier transform formula: $H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} \beta\delta(t-\tau) e^{-j2\pi f t} dt$

$y(t) = \beta x(t-\tau)$ $\xrightarrow{\mathcal{F}}$ $Y(f) = \beta e^{-j2\pi f \tau} X(f)$

$H(f) = \frac{Y(f)}{X(f)} =$

Distortionless Channel

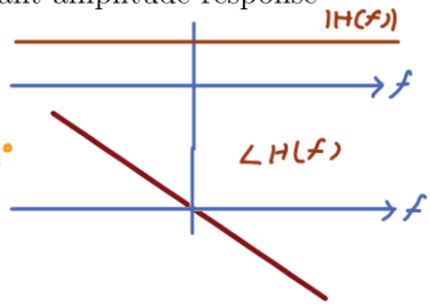
- A channel is distortionless if and only if it satisfies two properties:

  (a) "flat" frequency response: constant amplitude response

  $|H(f)| = |\beta|$

  (b) linear phase shift

  $\angle H(f) = -2\pi\tau f \pm m\,180^\circ$



3.19. Major types of distortion

(a) Linear distortion

(i) Amplitude distortion (frequency distortion): $H(f)$ is not constant with frequency.

$|H(f)| \neq |\beta|$

(ii) Phase distortion (delay distortion): the phase shift is not linear; the various frequency components suffer different amounts of time delay

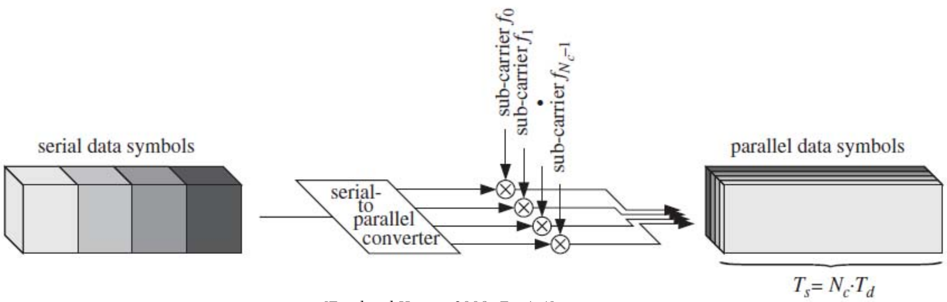
not satisfied.

Observation and a Solution

- Observation: Delay spread causes ISI
- A general rule of thumb is that a delay spread of less than 5 or 10 times the symbol width will not be a significant factor for ISI.
- Solution: The ISI can be mitigated by reducing the symbol rate and/or including sufficient guard times between symbols.

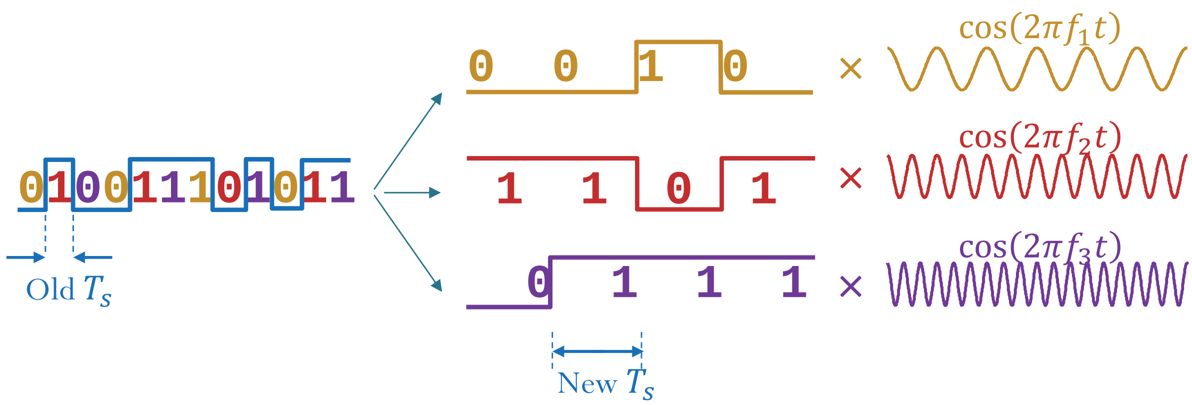
Multi-Carrier Transmission

- Convert a serial high rate data stream on to **multiple parallel low rate** sub-streams.
- Each sub-stream is modulated on its own **sub-carrier**.
- **Time domain perspective**: Since the symbol rate on each sub-carrier is much less than the initial serial data symbol rate, the effects of delay spread, i.e. ISI, significantly decrease, reducing the complexity of the equalizer.



[Fazel and Kaiser, 2008, Fig 1-4]

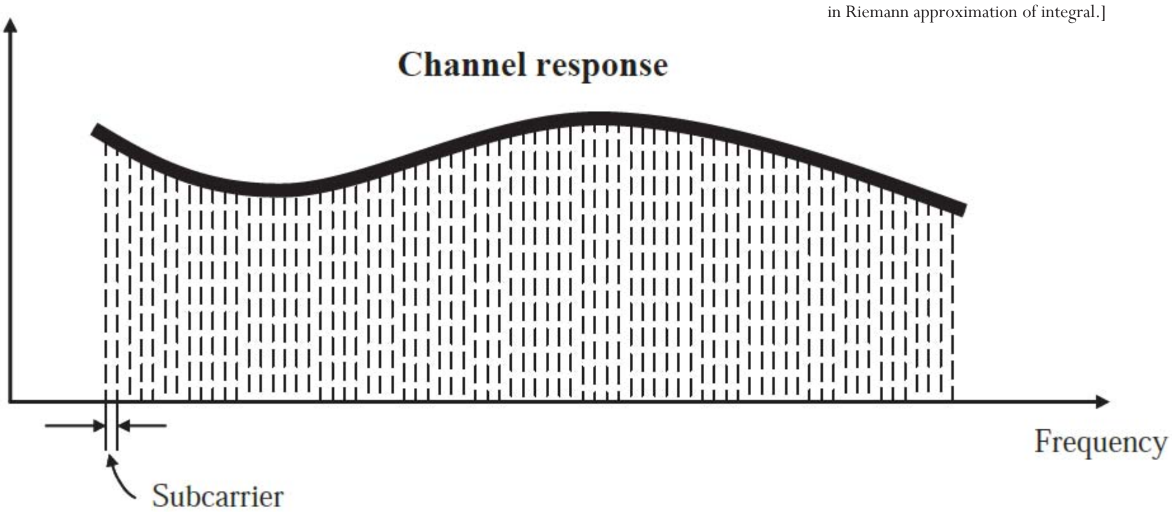
Multi-Carrier Modulation



Frequency Division Multiplexing

- **Frequency Domain Perspective**: Even though the fast fading is frequency-selective across the entire OFDM signal band, it is effectively flat in the band of each low-speed signal.

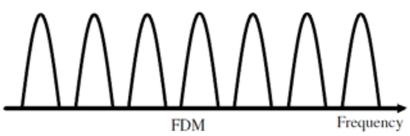
[The flatness assumption is the same one that you used in Riemann approximation of integral.]



[Myung and Goodman, 2008]

Frequency Division Multiplexing (FDM)

- To facilitate separation of the signals at the receiver, the carrier frequencies were **spaced sufficiently far apart** so that the signal spectra did not overlap. Empty spectral regions between the signals assured that they could be separated with readily realizable filters.
- The resulting spectral efficiency was therefore quite low.

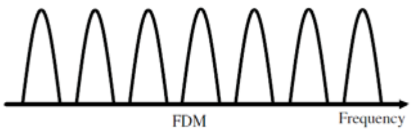


Single Carrier vs. Multi-Carrier (FDM)

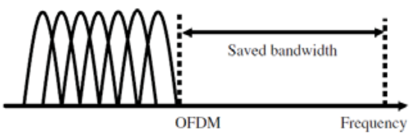
Single Carrier	Multi-Carrier (FDM)
Single higher rate serial scheme	Parallel scheme. Each of the parallel subchannels can carry a low signaling rate, proportional to its bandwidth.
❍ Multipath problem: Far more susceptible to inter-symbol interference (ISI) due to the short duration of its signal elements and the higher distortion produced by its wider frequency band ❍ Complicated equalization	❍ Long duration signal elements and narrow bandwidth in sub-channels. ❍ Complexity problem: If built straightforwardly as several (N) transmitters and receivers, will be more costly to implement. ❍ BW efficiency problem: The sum of parallel signalling rates is less than can be carried by a single serial channel of that combined bandwidth because of the unused guard space between the parallel sub-carriers.

OFDM

- **O**FDM = **Orthogonal** frequency division multiplexing
- One of multi-carrier modulation (MCM) techniques
  - Parallel data transmission (of many sequential streams)
  - A broadband is divided into many narrow sub-channels
  - Frequency division multiplexing (FDM)



- High spectral efficiency
  - The sub-channels are made **orthogonal** to each other over the **OFDM symbol duration** T_s.
    - Spacing is carefully selected.
  - Allow the sub-channels to overlap in the frequency domain.
  - Sub-carriers are spaced as close as theoretically possible.



Baseband OFDM Symbol

- Let $\underline{\mathbf{S}} = (S_1, S_2, \ldots, S_N)$ be the information vector.
- One baseband OFDM modulated symbol can be expressed as

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k t}{T_s} \right), \quad 0 \le t \le T_s$$

$$= \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp\left(j \frac{2\pi k t}{T_s} \right)}_{c_k(t)}$$

Some references may use different constant in the front

Some references may start with different time interval, e.g. $[-T_s/2, +T_s/2]$

Note that:

$$\mathrm{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\mathrm{Re}\{S_k\} \cos\left(\frac{2\pi k t}{T_s} \right) - \mathrm{Im}\{S_k\} \sin\left(\frac{2\pi k t}{T_s} \right) \right)$$

[ECS332 Section 4.6]

Review: QAM

Definition 4.76. In ***quadrature amplitude modulation*** (***QAM***) or ***quadrature multiplexing***, two baseband real-valued signals $m_1(t)$ and $m_2(t)$ are transmitted simultaneously via the corresponding QAM signal:

Form 1: $x_{\mathrm{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)$.

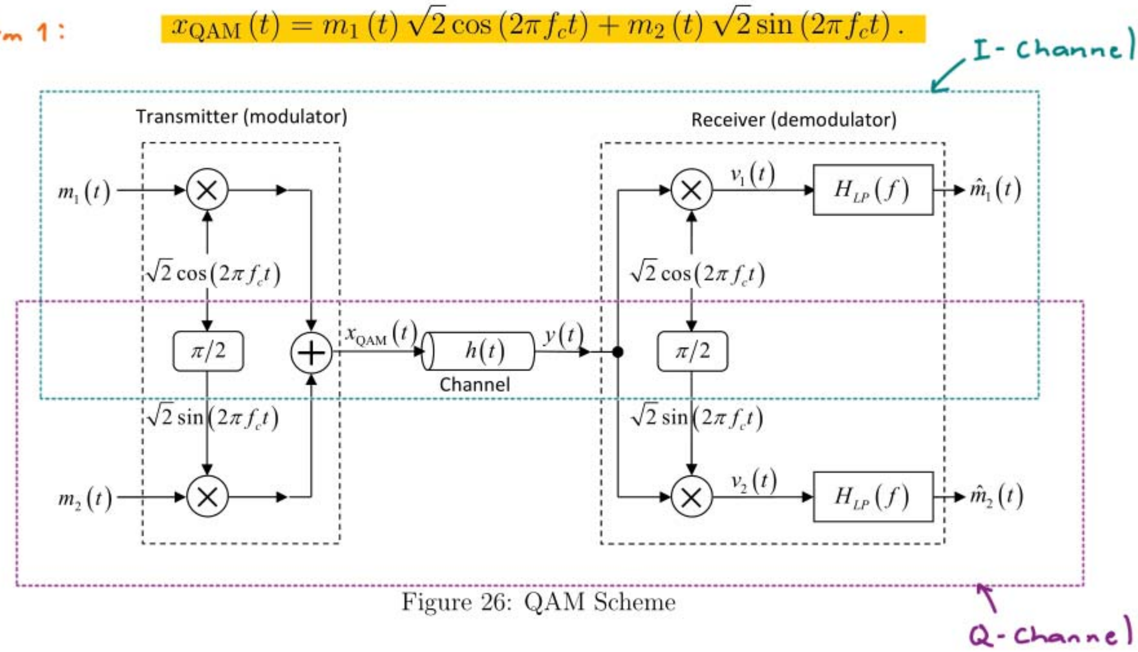


Figure 26: QAM Scheme

Review: QAM

4.81. Sinusoidal form (envelope-and-phase description [3, p. 165]):

Form #2

$$x_{\text{QAM}}(t) = \sqrt{2}E(t)\cos(2\pi f_c t + \phi(t)),$$

where

$$\text{envelope:} \quad E(t) = |m_1(t) - jm_2(t)| = \sqrt{m_1^2(t) + m_2^2(t)}$$

$$\text{phase:} \quad \phi(t) = \angle\left(m_1(t) - jm_2(t)\right)$$

calculator

75

rectangular → polar

$m_1(t) - jm_2(t) = E(t) \angle \phi(t)$

4.84. Complex form:

Form 3:

$$x_{\text{QAM}}(t) = \sqrt{2}\text{Re}\left\{(m(t))\, e^{j2\pi f_c t}\right\}$$

where[20] $m(t) = m_1(t) - jm_2(t)$.

OFDM and CDMA: Waveform Version

- Recall: Orthogonality-Based MA (CDMA)

$$s(t) = \sum_{k=0}^{\ell-1} S_k c_k(t) \quad \text{where} \quad c_{k_1} \perp c_{k_2}$$

- Baseband OFDM modulated symbol:

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kt}{T_s}\right), \quad 0 \le t \le T_s$$

$$= \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp\left(j\frac{2\pi kt}{T_s}\right)}_{c_k(t)}$$

Another "special case" of CDMA!

OFDM: Orthogonality

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \int_0^{T_s} \exp\left(j\frac{2\pi k_1 t}{T_s}\right)\exp\left(-j\frac{2\pi k_2 t}{T_s}\right)dt$$

$$= \int_0^{T_s} \exp\left(j\frac{2\pi(k_1-k_2)t}{T_s}\right)dt = \begin{cases} T_s, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases}$$

When $k_1 = k_2$,

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \int_0^{T_s} 1dt = T_s$$

When $k_1 \neq k_2$,

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \frac{T_s}{j2\pi(k_1-k_2)}\exp\left(j\frac{2\pi(k_1-k_2)t}{T_s}\right)\Bigg|_0^{T_s}$$

$$= \frac{T_s}{j2\pi(k_1-k_2)}(1-1) = 0$$

Frequency Spectrum

$$s(t) = \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t)\exp\left(j\frac{2\pi kt}{T_s}\right)}_{c_k(t)}$$

$$\Delta f = \frac{1}{T_s}$$

This is the term that makes the technique FDM.

$$1_{\left[-\frac{T_s}{2},\frac{T_s}{2}\right]}(t) \xrightarrow{\mathcal{F}} T_s \operatorname{sinc}(\pi T_s f)$$

$$c(t) = \frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \xrightarrow{\mathcal{F}} C(f) = \frac{1}{\sqrt{N}} T_s e^{-j2\pi f\frac{T_s}{2}} \operatorname{sinc}(\pi T_s f)$$

$$c_k(t) = c(t)\exp\left(j\frac{2\pi kt}{T_s}\right) \xrightarrow{\mathcal{F}} C_k(f) = C\left(f - \frac{k}{T_s}\right) = C(f - k\Delta f)$$

$$s(t) = \sum_{k=0}^{N-1} S_k c_k(t) \xrightarrow{\mathcal{F}} S(f) = \sum_{k=0}^{N-1} S_k C_k(f)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k e^{-j2\pi(f-k\Delta f)\frac{T_s}{2}} T_s \operatorname{sinc}(\pi T_s (f - k\Delta f))$$

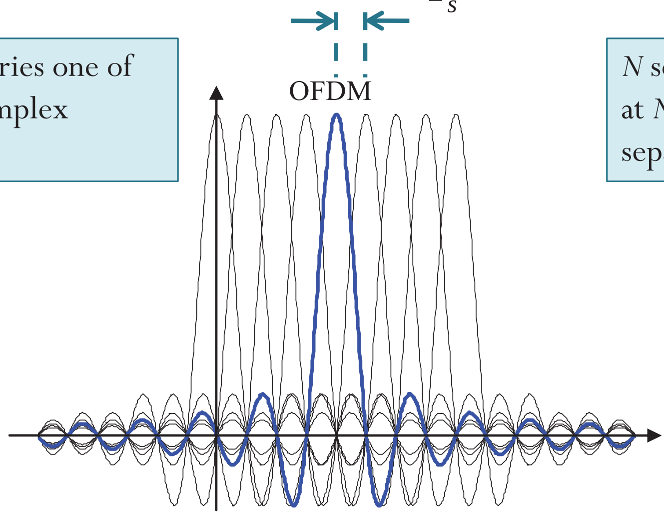
Subcarrier Spacing

$$s(t)=\sum_{k=0}^{N-1} S_k \frac{1}{\sqrt{N}} 1_{[0,T_s]}(t)\exp\left(j\frac{2\pi kt}{T_s}\right)$$

$$S(f)=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k e^{-j2\pi(f-k\Delta f)\frac{T_s}{2}} T_s \operatorname{sinc}\left(\pi T_s\left(f-k\Delta f\right)\right)$$

Each QAM signal carries one of the original input complex numbers.

$$\Delta f=\frac{1}{T_s}$$

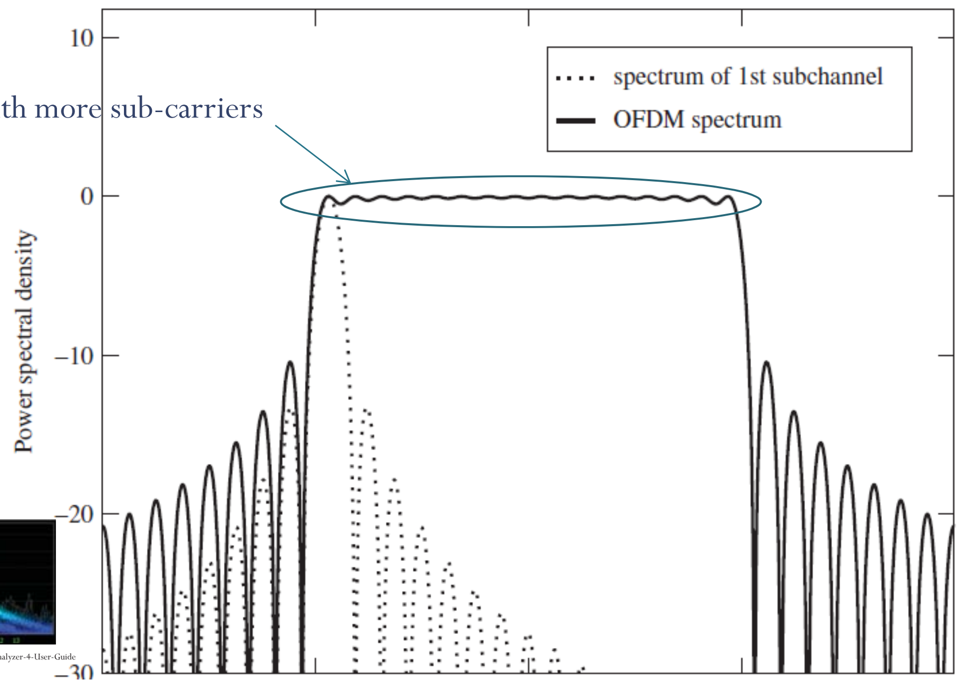


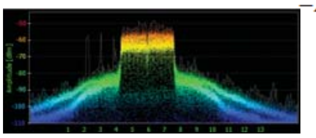
Spectrum Overlap in OFDM

N separate QAM signals, at *N* frequencies separated by the signaling rate.

The spectrum of each QAM signal is of the form with nulls at the center of the other sub-carriers.

Normalized Power Density Spectrum



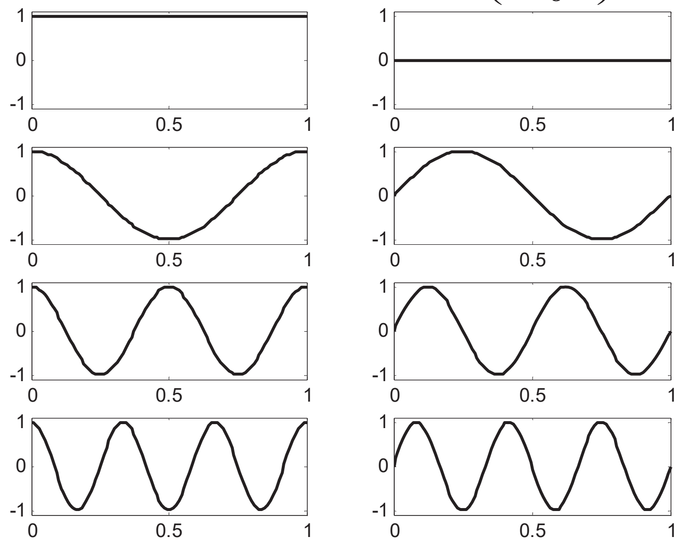


http://www.metageek.net/forums/showthread.php?4912-Chanalyzer-4-User-Guide

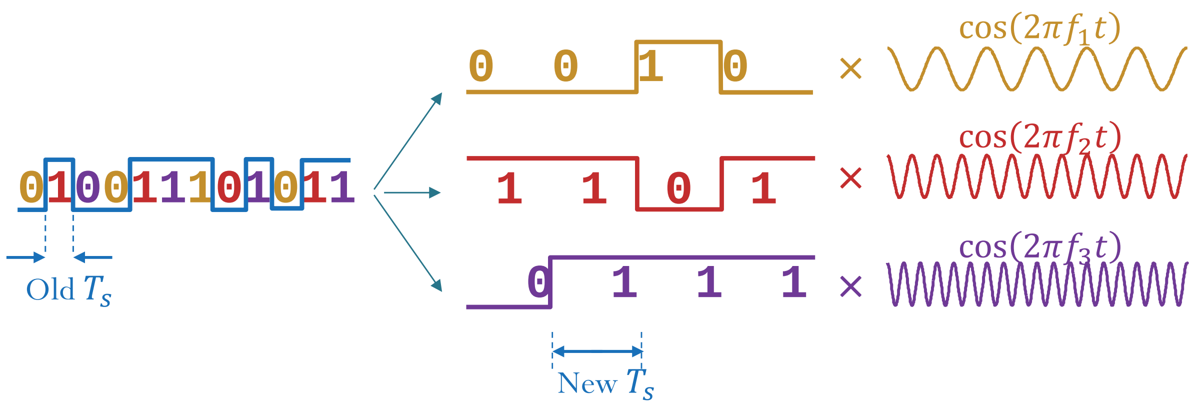
[Fazel and Kaiser, 2008, Fig 1-5]

OFDM Carriers: *N* = 4

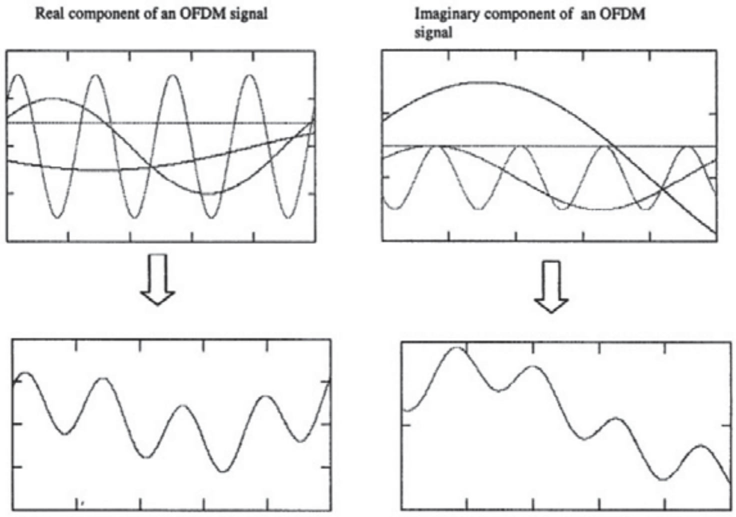
$$\cos\left(\frac{2\pi kt}{T_s}\right) \qquad \sin\left(\frac{2\pi kt}{T_s}\right)$$



Recall: Multi-Carrier Modulation



Time-Domain Signal



[Bahai, 2002, Fig 1.7]

Real and Imaginary components of an OFDM symbol is the *superposition of several harmonics* modulated by data symbols

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kt}{T_s} \right), \quad 0 \le t \le T_s$$

$$\mathrm{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\underbrace{\mathrm{Re}\{S_k\} \cos\left(\frac{2\pi kt}{T_s} \right)}_{\text{in-phase part}} - \underbrace{\mathrm{Im}\{S_k\} \sin\left(\frac{2\pi kt}{T_s} \right)}_{\text{quadrature part}} \right)$$

Summary

- So, we have a scheme which achieves
  - Large symbol duration (T_s) and hence less multipath problem
  - Good spectral efficiency
- One more problem:
  - There are so many carriers!

Discrete Fourier Transform (DFT)

Transmitter produces

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k}{T_s} t \right), \quad 0 \le t < T_s$$

Sample the signal **in time domain every T_s / N** gives

$$s[n] = s\left(n \frac{T_s}{N} \right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k}{\cancel{T_s}} n \frac{\cancel{T_s}}{N} \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k n}{N} \right) = \sqrt{N}\, \mathrm{IDFT}\{S\}[n]$$

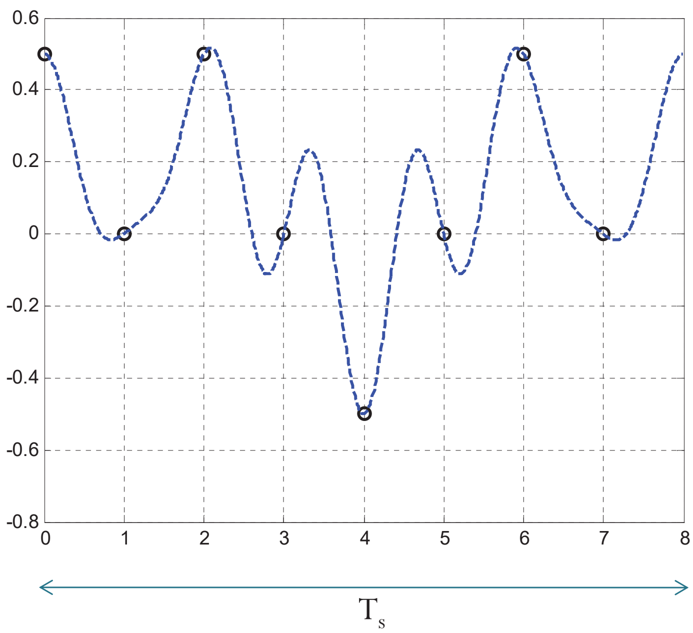
where $\mathrm{IDFT}\{\vec{\mathbf{S}}\}[n] = \frac{1}{N} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k n}{N} \right)$

$\vec{\mathbf{S}} = \left(S_0 \quad S_1 \quad \cdots \quad S_{N-1} \right)^T$

We can implement OFDM in the discrete domain!

DFT Samples

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k t}{T_s} \right), \quad 0 \le t \le T_s$$



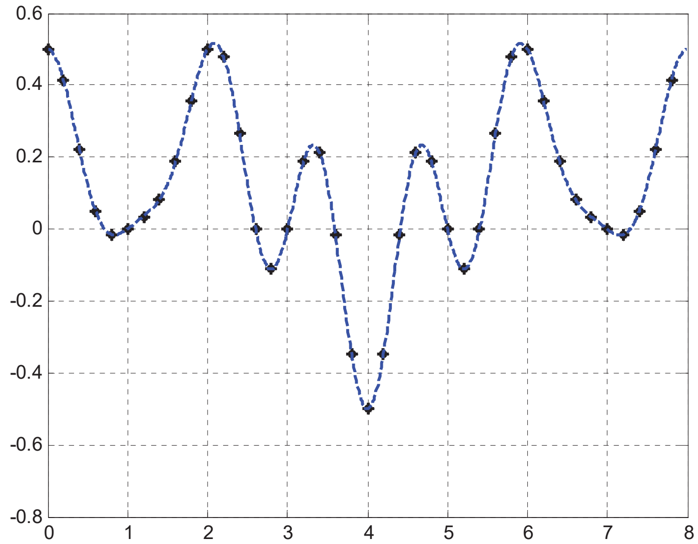
$$s[n] = s\left(n \frac{T_s}{N} \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k n}{N} \right)$$

$$= \sqrt{N}\, \mathrm{IDFT}\{S\}[n]$$

$$0 \le n < N$$

Oversampling



Oversampling (2)

- Increase the number of sample points from N to LN on the interval $[0, T_s]$.
- ***L*** is called the **over-sampling factor**.

$$s[n] = s\left(n\frac{T_s}{N}\right) \quad 0 \le n < N \quad \Rightarrow \quad s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) \quad 0 \le n < LN$$

$$s^{(L)}[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi k}{T_s} n \frac{T_s}{LN}\right) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)$$

$$= \frac{1}{\sqrt{N}} LN\left(\frac{1}{LN}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\left(\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right) + \sum_{k=N}^{NL-1} 0 \exp\left(j\frac{2\pi kn}{LN}\right)\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\sum_{k=0}^{NL-1} \tilde{S}_k \exp\left(j\frac{2\pi kn}{LN}\right)\right) = L\sqrt{N}\ \mathrm{IDFT}\{\tilde{S}\}[n]$$

Zero padding:

$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$

Scaling

Oversampling: Summary

N points

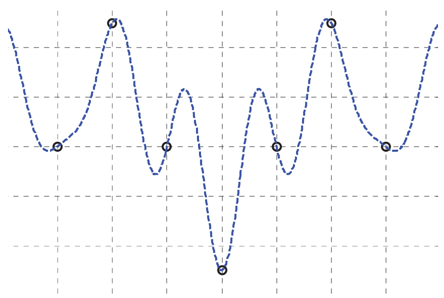
$$s[n] = s\left(n\frac{T_s}{N}\right) = \sqrt{N}\,\text{IDFT}\{\mathbf{S}\}[n]$$
$$0 \le n < N$$

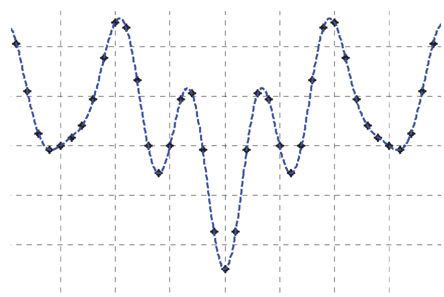
LN points

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) = L\sqrt{N}\,\text{IDFT}\{\tilde{\mathbf{S}}\}[n]$$
$$0 \le n < LN$$

Zero padding:

$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$





Summary: Three steps towards modern OFDM

1. To mitigate multipath problem
   → Use multicarrier modulation (FDM)
2. To gain spectral efficiency
   → Use orthogonality of the carriers
3. To achieve efficient implementation
   → Use FFT and IFFT